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COMMENT

Remarks on the finite temperature effect in supersymmetric quantum mechanics

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Abstract. Using the thermofield dynamics formalism we study the effect of temperature on supersymmetry within the context of supersymmetric quantum mechanics. The model considered here involves an interaction not of polynomial type and it is shown here that the finite-temperature effect causes spontaneous breaking of supersymmetry.

In recent times finite-temperature effects in supersymmetric theories have been studied in considerable detail. However, all the papers have dealt with supersymmetric field theories and only very recently have finite-temperature effects in supersymmetric quantum mechanics (SUSYQM) been studied (Das *et al* 1986, Fuchs 1985, Roy and Roychoudhury 1987a). In these papers it has been shown that supersymmetry is spontaneously broken at finite temperature. However, in these papers models based on superpotentials with polynomial character only were considered. In the present paper we shall study the finite-temperature effect on a SUSYQM model based on a superpotential with (non-singular) non-polynomial character. More precisely, we shall evaluate the temperature-dependent ground-state energy to study the symmetry behaviour. Throughout the calculations we shall use the thermofield dynamics formalism (Ojima 1981). We shall also indicate a general method of performing perturbative calculations at finite temperature when the interaction is of non-polynomial character.

To begin with, let us first specify the superpotential of the model (Roy and Roychoudhury 1987b, c):

$$W(x) = x + 2gx/(1+gx^2). \tag{1}$$

The SUSYQM Hamiltonian corresponding to this superpotential is given by (Cooper and Freedman 1983)

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}W^2(x) + \tfrac{1}{2}W'(x)[\bar{\psi}, \psi] \tag{2}$$

where x, p and ψ satisfy the following relations:

$$[x, p] = \mathrm{i} \qquad \{\bar{\psi}, \psi\} = 1 \tag{3}$$

and all other brackets vanish.

From (1) the zero-energy wavefunctions can be easily found and are given by

$$\varphi_{\pm}^{0}(x) \sim \exp\left(\pm\int^{x} W(t)\,\mathrm{d}t\right) \tag{4}$$

and since $\varphi_-^0(x) \sim (1+gx^2)^{-1} \exp(-\frac{1}{2}x^2) \to 0$ as $x \to \pm\infty$, we conclude that a physically acceptable ground state (of zero energy, i.e. $E_G = 0$) exists and so supersymmetry is unbroken.

Also from (1) and (2) we can write H in the form

$$H = H_0 + \tilde{V} \tag{5}$$

where H_0 denotes the Hamiltonian of the SUSY harmonic oscillator

$$H_0 = \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}[\bar{\psi}, \psi] \tag{6}$$

and $\tilde{V}$ denotes the interaction potential:

$$\tilde{V} = 2 + \frac{(g-2)}{(1+gx^2)} + \frac{2g}{(1+gx^2)}\psi\bar{\psi} - \frac{4g}{(1+gx^2)^2}\psi\bar{\psi}. \tag{7}$$

Next we introduce the bosonic and fermionic creation and annihilation operators:

$$a^+ = (p + \mathrm{i}x)/\sqrt{2} \qquad a = (p - \mathrm{i}x)/\sqrt{2} \tag{8}$$

$$b^+ = \bar{\psi} \qquad b = \psi. \tag{9}$$

Using (3), (8) and (9) the following relations can now be established:

$$[a, a^+] = \{b, b^+\} = 1 \qquad [a^+, a^+] = [a, a] = \{b^+, b^+\} = \{b, b\} = [a, b^+] = 0. \tag{10}$$

It is now possible to write the Hamiltonian of the SUSY oscillator in the following form:

$$H_0 = (a^+a + b^+b) = (N_B + N_F) \tag{11}$$

where N_B and N_F stand for bosonic and fermionic number operators with eigenvalues given respectively by

$$n_B = 0, 1, 2, \ldots \qquad n_F = 0, 1. \tag{12}$$

The ground state of the SUSY oscillator is given by

$$|0\rangle = |n_B = 0, n_F = 0\rangle \tag{13}$$

and the ground-state energy is

$$E_0 = \langle n_B = 0, n_F = 0|H|n_B = 0, n_F = 0\rangle = 0. \tag{14}$$

Note that for the ground state given by (13) we have

$$a|0\rangle = b|0\rangle = 0. \tag{15}$$

Next we turn to the description of the system at finite temperature. In the thermofield dynamics approach (Ojima 1981) the Hilbert space of states is doubled by the introduction of tilde states denoted by $|\tilde{n}_B, \tilde{n}_F\rangle$. There are creation and annihilation operators denoted respectively by $\tilde{a}^+$, $\tilde{b}^+$, $\tilde{a}$, $\tilde{b}$ acting on the tilde states. Also the tilde operators satisfy the same commutation and anticommutation relations as the non-tilde ones.

A physical state would now look like $|n_B, n_F\rangle \otimes |\tilde{n}_B, \tilde{n}_F\rangle$; thus, for example, the ground state of the SUSY oscillator would be $|0\rangle = |n_B = 0, n_F = 0\rangle \otimes |\tilde{n}_B = 0, \tilde{n}_F = 0\rangle$ and corresponding to (15) we have

$$\tilde{a}|0\rangle = \tilde{b}|0\rangle = 0. \tag{16}$$

It is now necessary to determine the forms of various creation and annihilation operators at finite temperature. These are given by (Ojima 1981)

$$a(\beta)=a\cosh\theta(\beta)-\tilde{a}^{+}\sinh\theta(\beta) \tag{17}$$

$$\tilde{a}(\beta)=\tilde{a}\cosh\theta(\beta)-a^{+}\sinh\theta(\beta) \tag{18}$$

where $\theta(\beta)$ is defined by

$$\cosh\theta(\beta)=(1-e^{-\beta})^{-1/2} \qquad \sinh\theta(\beta)=e^{-\beta/2}(1-e^{-\beta})^{-1/2}. \tag{19}$$

Fermion annihilation operators are given by

$$b(\beta)=b\cos\bar{\theta}(\beta)-\tilde{b}^{+}\sin\bar{\theta}(\beta) \tag{20}$$

$$\tilde{b}(\beta)=\tilde{b}\cos\bar{\theta}(\beta)+b^{+}\sin\bar{\theta}(\beta) \tag{21}$$

where $\bar{\theta}(\beta)$ is defined by

$$\cos\bar{\theta}(\beta)=(1+e^{-\beta})^{-1/2} \qquad \sin\bar{\theta}(\beta)=e^{-\beta/2}(1+e^{-\beta})^{-1/2}. \tag{22}$$

The creation operators can be obtained by taking Hermitian conjugates of the annihilation operators:

$$a^{+}(\beta)=a^{+}\cosh\theta(\beta)-\tilde{a}\sinh\theta(\beta) \tag{23}$$

$$\tilde{a}^{+}(\beta)=\tilde{a}^{+}\cosh\theta(\beta)-a\sinh\theta(\beta) \tag{24}$$

$$b^{+}(\beta)=b^{+}\cos\bar{\theta}(\beta)-\tilde{b}\sin\bar{\theta}(\beta) \tag{25}$$

$$\tilde{b}^{+}(\beta)=\tilde{b}^{+}\cos\bar{\theta}(\beta)+b\sin\bar{\theta}(\beta). \tag{26}$$

It may be pointed out that the relations (17), (18), (20), (21) and (23)-(26) may be inverted, so that the zero-temperature operators can be expressed in terms of the temperature-dependent ones, for example,

$$a=a(\beta)\cosh\theta(\beta)+\tilde{a}^{+}(\beta)\sinh\theta(\beta) \tag{27}$$

$$b=b(\beta)\cos\bar{\theta}(\beta)+\tilde{b}^{+}(\beta)\sin\bar{\theta}(\beta). \tag{28}$$

The temperature-dependent operators satisfy the following relations:

$$[a(\beta), a^{+}(\beta)]=[\tilde{a}(\beta), \tilde{a}^{+}(\beta)]=\{b(\beta), b^{+}(\beta)\}=\{\tilde{b}(\beta), \tilde{b}^{+}(\beta)\}=1 \tag{29}$$

and all other brackets vanish.

We note that the thermal vacuum is given by

$$|O(\beta)\rangle=|n_{B}(\beta)=0, n_{F}(\beta)=0\rangle\otimes|\tilde{n}_{B}(\beta)=0, \tilde{n}_{F}(\beta)=0\rangle \tag{30}$$

and it is annihilated by the destruction operators:

$$a(\beta)|O(\beta)\rangle=\tilde{a}(\beta)|O(\beta)\rangle=b(\beta)|O(\beta)\rangle=\tilde{b}(\beta)|O(\beta)\rangle=0. \tag{31}$$

Also we have

$$\langle O(\beta)|O(\beta)\rangle=1. \tag{32}$$

An important point to note is that the temperature-dependent vacuum expectation value of any operator X can be calculated from the following relation (Ojima 1981):

$$\langle X\rangle_{\beta}=\langle O(\beta)|X|O(\beta)\rangle. \tag{33}$$

To determine the effect of temperature on supersymmetry it is now necessary to evaluate the temperature-dependent ground-state energy of the system specified by (1) (it may be recalled here that in SUSY the ground-state energy serves as an order parameter). We shall perform this by using first-order perturbation theory. In other words we have to find the following:

$$E_G(\beta)=\langle O(\beta)|H|O(\beta)\rangle=\langle O(\beta)|(H_0+\tilde{V})|O(\beta)\rangle$$
$$=\langle O(\beta)|H_0|O(\beta)\rangle+\langle O(\beta)|\tilde{V}|O(\beta)\rangle. \tag{34}$$

Using (31), (32) and their Hermitian conjugates we find from (11) that

$$\langle O(\beta)|H_0|O(\beta)\rangle=\left(\frac{e^{-\beta}}{1+e^{-\beta}}+\frac{e^{-\beta}}{1-e^{-\beta}}\right). \tag{35}$$

Before evaluating the second expression in (34) we note that from relation (8) we have

$$x^2=-\tfrac{1}{2}(a^+-a)^2 \tag{36}$$

and using (31) we can write this as

$$x^2=-\tfrac{1}{2}(A^+-A)^2 \tag{37}$$

where the operators A and A^+ are given by

$$A=a(\beta)\cosh\theta(\beta)-\tilde{a}(\beta)\sinh\theta(\beta) \tag{38}$$

$$A^+=a^+(\beta)\cosh\theta(\beta)-\tilde{a}^+(\beta)\sinh\theta(\beta). \tag{39}$$

The operators A and A^+ satisfy the following commutation relation:

$$[A,A^+]=1. \tag{40}$$

Also we have

$$A|O(\beta)\rangle=0. \tag{41}$$

The second term in the second expression of (38) can now be evaluated as follows:

$$\left\langle O(\beta)\left|\frac{(g-2)}{(1+gx^2)}\right|O(\beta)\right\rangle=(g-2)\int_0^\infty d\alpha\,\exp(-\alpha)\langle O(\beta)|\exp[\tfrac{1}{2}\alpha g(A^+-A)^2]|O(\beta)\rangle$$
$$=(g-2)\int_0^\infty d\alpha\,\exp(-\alpha)\left\langle O(\beta)\left|\sum_{n=0}^{\infty}\frac{1}{n!}(\tfrac{1}{2}\alpha g)^n(A^+-A)^{2n}\right|O(\beta)\right\rangle. \tag{42}$$

Now using the identity

$$A(A^+-A)^n=n(A^+-A)^{n-1}+(A^+-A)^nA \tag{43}$$

repeatedly in the operator product expansion on the RHS of (42) we get

$$\left\langle O(\beta)\left|\frac{(g-2)}{(1+gx^2)}\right|O(\beta)\right\rangle=(g-2)\sum_{n=0}^{\infty}(-1)^n\frac{g^n}{\sqrt{\pi}}\Gamma(n+\tfrac{1}{2}). \tag{44}$$

The sum in (44) can be performed by the Borel summation technique. For the sake of completeness we quote the result (Popov *et al* 1977).

If $f(z)$ is defined by

$$f(z)=\sum_{k=k_0}a_k(-z)^k \tag{45}$$

with

$$a_k = \frac{(k\alpha)!}{k!} a^k \sum_{m=0}^{\infty} c_m \Gamma(k+\beta-m+1) \tag{46}$$

then the sum of the series is given by

$$f(z) = (az)^{-(\beta+1)} e^{1/az} \sum_{m=0}^{\infty} \Gamma(\beta-m+1) c_m (az)^m F(az, \alpha, m-\beta) \tag{47}$$

where

$$F(x, \alpha, \gamma) = x^{-\gamma} \int_1^{\infty} e^{-t/x} [(t-1)^{\alpha}+1]^{\gamma-1} \, dt. \tag{48}$$

Using (45)-(48) in (47) we get

$$\left\langle O(\beta) \left| \frac{1}{(1+gx^2)} \right| O(\beta) \right\rangle = (g-2) g^{-1/2} e^{1/g} \Gamma(\tfrac{1}{2}, 1/g) \tag{49}$$

where $\Gamma(a, x)$ denotes the incomplete gamma function (Gradshteyn and Ryzhik 1980).

Proceeding similarly we find for the third term in the second expression of (35)

$$\left\langle O(\beta) \left| \frac{2g}{(1+gx^2)} bb^+ \right| O(\beta) \right\rangle = 2g(g)^{-1/2} e^{1/g} \Gamma(\tfrac{1}{2}, 1/g) \cos^2 \bar{\theta}(\beta). \tag{50}$$

Now differentiating both sides of (49) WRT g the fourth term is found to be

$$\left\langle O(\beta) \left| \frac{4g}{(1+gx^2)^2} bb^+ \right| O(\beta) \right\rangle = 2g^{-1/2} e^{1/g} [\Gamma(-\tfrac{1}{2}, \tfrac{1}{2}) + g\Gamma(\tfrac{1}{2}, 1/g)] \cos^2 \bar{\theta}(\beta). \tag{51}$$

Finally, collecting all the terms we get

$$\begin{aligned} E_G(\beta) &= \langle O(\beta) | H | O(\beta) \rangle \\ &= 2 + (g-2)(g)^{-1/2} e^{1/g} \Gamma(\tfrac{1}{2}, 1/g) + \left(\frac{e^{-\beta}}{1-e^{-\beta}} + \frac{e^{-\beta}}{1+e^{-\beta}} \right) \\ &\quad - 2(g)^{-1/2} e^{1/g} \Gamma(-\tfrac{1}{2}, 1/g) \left(\frac{1}{1+e^{-\beta}} \right) \\ &\sim \frac{(2g+1) e^{-\beta}}{(1+e^{-\beta})} + \frac{e^{-\beta}}{(1-e^{-\beta})} + O(g^2). \end{aligned} \tag{52}$$

Since $E_G(\beta) > 0$ for any β we conclude that supersymmetry is spontaneously broken. Note that for $\beta \to \infty$ we recover the correct zero-temperature behaviour, namely $E_G(T=0) = 0$ (in the first order in g) indicating that SUSY is unbroken at zero temperature.

In conclusion it has been shown here that the finite-temperature effect causes spontaneous breakdown of supersymmetry. This result supports an earlier conclusion of Das *et al* (1986). Also we have outlined a general method of performing perturbative calculations at finite temperature when the interaction is not of polynomial character.

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